Winter term 2024/25
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Presence sheet 03

Mathematics for Machine Learning

Tutorial of **Week 04 (04.11. - 08.11.2024)**

Exercise 1 (Scalar product).
Consider the vector space $V = \mathcal{C}([a,b])$ for $a < b$. Prove that the following function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ with $\langle f, g \rangle = \int_a^b f(t)g(t)\mathrm{d}t$ defines a scalar product.

Exercise 2 (Orthonormal basis).
Decide if the following function sets of vectors are orthonormal bases of $\mathbb{R}^3$ with respect to the standard scalar product:

a) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \subset \mathbb{R}^3$

b) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ -2 \end{pmatrix} \right\} \subset \mathbb{R}^3$

c) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix}, \begin{pmatrix} 0 \\ \sqrt{2}/2 \\ -\sqrt{2}/2 \end{pmatrix} \right\} \subset \mathbb{R}^3$

Exercise 3 (Projection).
Decide if the following matrices are projections.

a) $A = \begin{pmatrix} 0 & 0 \\ 7 & 1 \end{pmatrix}$

b) $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

c) $C = \begin{pmatrix} 1 & 0 \\ 5 & 4 \end{pmatrix}$

Which of these is an orthogonal projection?

Exercise 4 (Orthogonal matrix).
Consider an orthogonal matrix Q. Prove the following properties for the standard scalar product:

a) $\langle Qv, Qw \rangle = \langle v, w \rangle$

b) $\|Qv\| = \|v\|$.

c) $|\det(Q)| = 1$.

Exercise 5 (Positive definite matrices).

Consider the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -6 \end{pmatrix}$.

a) Find the eigenvalues of A and their algebraic multiplicity.

b) Decide whether A is positive (semi-)definite.

c) Decide whether A is invertible.

d) Find a matrix that is not positive semidefinite but has only positive entries.

Exercise 6 (Diagonalizable matrices).
Find a matrix over $\mathbb{C}$ which is not diagonalizable.