Winter term 2024/25
U. von Luxburg
E. Günther/ K. Frohnapfel

Assignment 02

Mathematics for Machine Learning

Submission due Friday **01.11.24, 12:00** via Ilias

Justify all your claims.

Exercise 1 (**Change of Basis**, 2 + 2 + 1 points)**.**
Consider the linear map $T \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^3)$ with $T(x) = (-x_1, x_2, 2x_3)^T$ for $x = (x_1, x_2, x_3)^T \in \mathbb{R}^3$. Consider the standard basis $\mathcal{B} = \{e_1, e_2, e_3\}$ and the basis $\mathcal{C} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$ of $\mathbb{R}^3$.

a) Find the matrix $M(T, \mathcal{B}, \mathcal{B})$ which corresponds to the linear map T.

b) Find the transformation matrices $M(\mathrm{Id}, \mathcal{B}, \mathcal{C})$ and $M(\mathrm{Id}, \mathcal{C}, \mathcal{B})$.

c) Find the matrix $M(T, \mathcal{C}, \mathcal{C})$.

Exercise 2 (**Matrices**, 1+1+1+1+1 points)**.**
Consider the differentiation operator $D = \mathrm{d}/\mathrm{d}t \colon \mathbb{R}^\mathbb{R} \to \mathbb{R}^\mathbb{R}, f \mapsto f'$ on the vector space $\mathbb{R}^\mathbb{R}$ of all real functions. Below we give different choices of bases $\mathcal{W}$. For each of them, we consider the corresponding subspace $\mathcal{U} := \mathrm{span}(\mathcal{W})$ and the restricted linear map $D|_\mathcal{U} \colon \mathcal{U} \to \mathbb{R}^\mathbb{R}$, which is the differentiation operator just applied to vectors in $\mathcal{U}$. Decide whether $\mathrm{range}(D|_\mathcal{U}) \subseteq \mathcal{U}$ and if so, state the matrix $\mathcal{M}(D|_\mathcal{U}, \mathcal{W}, \mathcal{W})$.

a) $\mathcal{W} = \left\{e^t, e^{2t}\right\}$

b) $\mathcal{W} = \left\{1, t^2, t^4\right\}$

c) $\mathcal{W} = \left\{e^t, te^t\right\}$

d) $\mathcal{W} = \{\sin t, \cos t\}$

e) $\mathcal{W} = \left\{t, (\sin t)^2, (\cos t)^2, \sin t \cos t\right\}$

Exercise 3 (**Eigenvalues**, 2+3 points)**.**

a) Let $A \in \mathbb{R}^{n \times n}$ with $A^k = 0$ for some $k \in \mathbb{N}$. Prove that, if λ is an eigenvalue of A, then $\lambda = 0$.

b) Let V be a finite-dimensional vector space and $T \colon V \to V$ a linear map such that every $v \in V$ with $v \neq 0$ is an eigenvector of T. Prove that $T = \lambda \,\mathrm{Id}$ for some $\lambda \in \mathbb{R}$.

Exercise 4 (**Power Method**, 1+4 points)**.**
Let $A \in \mathbb{R}^{n \times n}$ be a diagonalizable matrix with one unique largest eigenvalue, that is, $|\lambda_1| > |\lambda_2| > ... > |\lambda_n|$, where λ_i are the eigenvalues. We furthermore assume $\lambda_1 > 0$.
We consider the *power method*, a method to numerically estimate an eigenvector to the largest eigenvalue λ_1. For an arbitrary initial vector $x_0 \in \mathbb{R}^n$ we recursively define

$$x_{k+1} := \frac{Ax_k}{||Ax_k||}.$$

a) Prove that $x_k = \frac{A^k x_0}{||A^k x_0||}$.

b) Consider a basis of eigenvectors $v_1, ..., v_n$, where v_i belongs to λ_i, and the representation

$$x_0 = c_1 v_1 + ... + c_n v_n.$$

Prove that, if $c_1 \neq 0$, the sequence x_k converges to an eigenvector of λ_1 for $k \to \infty$.