Winter term 2024/25
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# Presence sheet 11

## Mathematics for Machine Learning

Tutorial of **Week 12 (13.01. - 17.01.2025)**

**Exercise 1 (The black sheep).**
A shepard has one black sheep. Each day they buy new white sheep, add them to the herd and afterwards pick a random sheep from the herd uniformly and independently from the previous days and shear it.

a) Assuming one white sheep is bought every day, what is the probability of the black sheep to be shorn infinitely many times?

b) Assuming $n$ white sheep are bought on day $n$, what is the probability of the black sheep to be shorn infinitely many times?

**Exercise 2 (Borel-Cantelli).**
In the second part of the Borel-Cantelli Lemma, the events are asked to be independent. Find a counter-example to prove that this assumption is indeed necessary.

**Exercise 3 (Bernstein inequality).**
Let $X_1, \dots, X_n$ be independent random variables with zero mean and $|X_i| < 1$ for all $i \in \{1, \dots, n\}$. Define $\sigma^2 := \frac{1}{n}\sum_{i=1}^n \mathrm{Var}(X_i)$ and $S_n := \frac{1}{n}\sum_{i=1}^n X_i$.

a) For $t > 0$ show that

$$P\left(|S_n| > t\right) \leq 2\exp\left(-\frac{nt^2}{2(\sigma^2 + \frac{t}{3})}\right).$$

b) Derive a lower bound for $n$ that guarantees

$$P\left(|S_n| > t\right) \leq \delta$$

for $t > 0$ and $0 < \delta \leq 1$.

**Exercise 4 (Continuous-Mapping Theorem).**
Consider a sequence of $d$-variate random vectors $X_n$ that converges almost surely to a random vector $X$ and a measurable function $f : \mathbb{R}^d \to \mathbb{R}^k$. Let $D_f = \{x \in \mathbb{R}^d | f \text{ is continuous at } x\}$ with $P\left(X \in D_f^C\right) = 0$. Then, $f(X_n) \to f(X)$ almost surely.