Winter term 2024/25
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Presence sheet 09

Mathematics for Machine Learning

Tutorial of **Week 10 (16.12. - 20.12.2024)**

Exercise 1 (σ-Algebra).

a) Consider the set $\Omega = \{1, 2, 3, 4\}$. Is the following set $\mathcal{F} \subset \mathcal{P}(\Omega)$ a σ-algebra on Ω?

$$\mathcal{F} = \{\emptyset, \{3\}, \{1, 2\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 3, 4\}\}$$

b) Let Ω be a non-empty set and $A \subseteq \Omega$. Compute the σ-algebra $\sigma(A)$ generated by A.

c) Find all σ-Algebras on $\Omega = \{0, 1, 2\}$.

Exercise 2 (Joint, marginal, conditional distribution).
Consider two random variables X and Y, where $X \in \mathcal{X} = \{\text{Sun}, \text{Rain}, \text{Snow}\}$ describes the weather and $Y \in \mathcal{Y} = \{\text{Few}, \text{Many}\}$ describes how many pedestrians are taking a stroll. The joint distribution of X and Y is given by the following table, which contains the probabilities $P(X = x, Y = y)$:

	Few	Many
Sun	0.1	0.4
Rain	0.27	0.03
Snow	0.06	0.14

a) Compute the probabilities of the following events:

- Many pedestrians on a sunny day or few pedestrians on a rainy day.
- The sun shines.
- The sun shines given that many pedestrians take a stroll.

b) Are X and Y independent?

Exercise 3 (Lebesgue-measure).
Consider the measure space $(\mathbb{R}, \mathcal{B}, \lambda)$ where $\mathcal{B}$ is the Borel-σ-Algebra on $\mathbb{R}$ and $\lambda : \mathcal{B} \to \mathbb{R}$ denotes the Lebesgue measure. Find a function $f : \mathbb{R} \to \mathbb{R}$ with $f \neq 0$ that is almost surely zero with respect to the Lebesgue measure.

Exercise 4 (Variance).
Show that $\mathrm{Var}(X) = \mathrm{E}\left(X^2\right) - \mathrm{E}(X)^2$.

Exercise 5 (Uniform distribution).
Consider the measure space $(\Omega, \mathcal{P}(\Omega), P)$ with $\Omega = \{1, \ldots, n\}$ and $P(\{i\}) = \frac{1}{n}$.

a) Show that $(\Omega, \mathcal{P}(\Omega), P)$ is a probability space.

b) How does the cummulative distribution function (cdf) look like?