Winter term 2024/25
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Presence sheet 04

Mathematics for Machine Learning

Tutorial of **Week 05 (11.11. - 15.11.2024)**

Exercise 1 (Matrix norms).
Consider the matrix $A = \begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}$ and calculate the following norms:

a) Supremum norm: $\|A\|_{\max}$

b) Frobenius norm: $\|A\|_F$

c) Spectral norm: $\|A\|_2$

d) Nuclear norm: $\|A\|_*$

Exercise 2 (Frobenius norm).
Consider the Frobenius norm $\|A\|_F = \sqrt{\sum_{i,j} a_{ij}^2}$.

a) Show that this is a norm on the space $\mathbb{R}^{n \times n}$.

b) Show that $\|A\|_F = \sqrt{tr(A^T A)}$.

Exercise 3 (Singular value decomposition).
Find the singular value decomposition of $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

Exercise 4 (Examples).
Find an example of (or explain why such an example cannot exist)

a) A square matrix with zero as its only eigenvalue and 5 as its only singular value.

b) A square matrix with zero as its only eigenvalue and singular values 0 and 5.

c) A square matrix $A \neq 0$ whose singular values all equal zero.

Exercise 5 (Singular values of the derivative).
Consider the vector space $P^2(\mathbb{R}) := \{ax^2 + bx + c \mid a, b, c \in \mathbb{R}\}$ of all polynomials of degree ≤ 2 and the linear map $D : P^2(\mathbb{R}) \to P^2(\mathbb{R})$, mapping a polynomial to its derivative. We again consider the scalar product $\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\mathrm{d}x$ from assignment 3.

a) For the orthonormal basis $\left(\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}x, \frac{3\sqrt{10}}{4}x^2 - \frac{\sqrt{10}}{4}\right)$ that we computed in assignment 3, compute the representing matrix of D.

b) Compute the singular values of the representation matrix of D.

c) If we choose a different orthonormal basis, do the singular values change?