Winter term 2024/25
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Presence sheet 01

Mathematics for Machine Learning

Tutorial of **Week 02 (21. - 25.10.2024)**

Exercise 1 (Diagonal Matrix).
Consider the diagonal matrix

$$A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}. \tag{1}$$

1. Calculate A^2.
2. Calculate A^n for any $n \in \mathbb{N} \setminus \{0\}$.

Exercise 2 (Basis).
Consider the following sets of vectors in $\mathbb{R}^3$. Decide which of them are bases of $\mathbb{R}^3$.

1. $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \subset \mathbb{R}^3$
2. $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \right\} \subset \mathbb{R}^3$
3. $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\} \subset \mathbb{R}^3$
4. $\left\{ \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\} \subset \mathbb{R}^3$
5. $\left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\} \subset \mathbb{R}^3$

Exercise 3 (Basis and Dimension).
Consider the subset $S := \{(x, y, z) \in \mathbb{R}^3 | x + y + z = 0\} \subset \mathbb{R}^3$.

1. Is S a subspace of $\mathbb{R}^3$?
2. Find a basis of S.
3. What is the dimension of this subspace?

Exercise 4 (Linear maps).
Decide whether the following maps are linear:

1. $T : \mathbb{R} \to \mathbb{R}$ with $T(x) = x^2$
2. $T : \mathbb{R}^3 \to \mathbb{R}$ with $T((x, y, z)^T) = 3x - y + z$
3. $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $T((x, y)^T) = (x + y, x - y)^T$

Exercise 5 (Linear maps and matrices).
Consider the two bases $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ and $\mathcal{C} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right\}$ of the vector space $\mathbb{R}^2$ and the identity mapping $Id : \mathbb{R}^2 \to \mathbb{R}^2$ with $Id(v) = v$. Find the following matrices:

1. $M(Id, \mathcal{B}, \mathcal{B})$
2. $M(Id, \mathcal{C}, \mathcal{C})$
3. $M(Id, \mathcal{B}, \mathcal{C})$

Exercise 6 (Linear maps and matrices).
Consider three vector spaces V_1, V_2 and V_3 of different dimension, that is, $dim(V_1) = n$, $dim(V_2) = m$, $dim(V_3) = k$ and $n \neq m \neq k$.
Let $\mathcal{B}_1$ be a basis of V_1, $\mathcal{B}_2$ a basis of V_2 and $\mathcal{B}_3$ a basis of V_3. Additionally consider another basis $\tilde{\mathcal{B}}_2$, which is a basis of V_2.
Now consider two linear maps $S : V_1 \to V_2$ and $T : V_2 \to V_3$.
Discuss whether the following equations hold:

1. $M(T \circ S, \mathcal{B}_1, \mathcal{B}_3) = M(T, \mathcal{B}_2, \mathcal{B}_3) \cdot M(S, \mathcal{B}_1, \mathcal{B}_2)$
2. $M(T \circ S, \mathcal{B}_1, \mathcal{B}_3) = M(T, \tilde{\mathcal{B}}_2, \mathcal{B}_3) \cdot M(S, \mathcal{B}_1, \mathcal{B}_2)$