Student's exam questions

1 Linear Algebra

Exercise 1 (LU decomposition (*))
Let $A = \begin{pmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{pmatrix} \in \mathbb{R}^{3x3}$

1. Calculate a lower triangular matrix $L \in \mathbb{R}^{3x3}$ and an upper triangular matrix $U \in \mathbb{R}^{3x3}$, so that $A = L \cdot U$
2. Calculate the determinant of matrix A.
3. Calculate the inverse of matrix A.

Exercise 2 ((**))
Consider a 3×2 matrix A where $A^T A = \begin{pmatrix} 10 & -8 \\ -8 & 10 \end{pmatrix}$. Using the properties of singular value decomposition, find the minimum possible value of $\|Ax\|$ where x is any unit vector in $\mathbb{R}^2$ (i.e., $x \in \mathbb{R}^2$ with $\|x\| = 1$), where $\|\cdot\|$ denotes the Euclidean norm. Justify your answer.

Exercise 3 (Cool exercise (**))
Let $\{x_1, \ldots, x_n\}$ be a set of vectors of a vector space V over a field F. For given coefficients $a_{i,j} \in F$ where $1 \leq i < j \leq n$, define:

$$y_j = x_j + \sum_{i<j} a_{i,j} x_i, \quad \text{for } j = 1, \ldots, n.$$

The transformation of the vectors can be expressed as a Matrix product:

$$\begin{bmatrix} y_1^T \\ y_2^T \\ \vdots \\ y_n^T \end{bmatrix} = A \begin{bmatrix} x_1^T \\ x_2^T \\ \vdots \\ x_n^T \end{bmatrix}$$

Prove that A is invertible.

Exercise 4 (Determinant (*))
Consider the matrix $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 2 \\ 3 & 0 & 1 \end{bmatrix}$. Is A invertible?

Exercise 5 (SVD and Matrix Norms (**))

a) Calculate the singular values of the matrix $A = \begin{pmatrix} -1 & 2 & 0 \\ 2 & 0 & -2 \\ 0 & -2 & 1 \end{pmatrix}$ and calculate the Frobenius- and the Spectral Norm of A.

b) Show that the condition $||A||_2 \leq ||A||_F \leq \sqrt{\min(n,m)}||A||_2$ holds for all matrices $A \in \mathbb{R}^{n \times m}$.

Exercise 6 (Rank of a Matrix (*))
Compute the rank of the following matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 1 \end{bmatrix}.$$

Exercise 7 (Eigenvalues of Projections (*))
Prove that any projection $P \in \mathcal{L}(\mathbb{R}^N)$ can only have eigenvalues 0 and 1.

Exercise 8 (Eigenvalues and Eigenvectors (*))
Let $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$. Find the eigenvalues and eigenvectors of A.

Exercise 9 (Properties of linear transformations(*))
For each of the following functions, determine if the function is a linear transformation. Remember to justify your reasoning and answers.

a) $f_1 : \mathbb{P} \to \mathbb{R}$ where $f_1(\vec{p}) =$ the degree of the polynomial $\vec{p}$.

b) $f_2 : \mathbb{P} \to \mathbb{R}$ where $f_2(\vec{p}) = \vec{p}(t = 1)$.

c) $f_3 : \mathbb{R}^2 \to \mathbb{R}^3$ where

$$f_3 \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a+b \\ a-b \\ b+1 \end{bmatrix}.$$

d) $f_4 : \mathbb{R}^3 \to \mathbb{R}^2$ where

$$f_4 \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a+b \\ a-c \end{bmatrix}.$$

e) $f_5 : \mathbb{R}^3 \to \mathbb{R}^2$ where

$$f_5 \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a+b \\ c^2 \end{bmatrix}.$$

Exercise 10 (SVD (***))
Suppose $T \in \mathcal{L}(V, W)$ and the positive singular values of T are $s_1, \ldots, s_m$. Suppose $e_1, \ldots, e_m$ and $f_1, \ldots, f_m$ are orthonormal lists in V and W such that

$$Tv = s_1\langle v, e_1\rangle f_1 + \cdots + s_m\langle v, e_m\rangle f_m \tag{1}$$

for every $v \in V$. Then show that

$$T^*w = s_1\langle w, f_1\rangle e_1 + \cdots + s_m\langle w, f_m\rangle e_m$$

and

$$T^\dagger w = \frac{\langle w, f_1\rangle}{s_1} e_1 + \cdots + \frac{\langle w, f_m\rangle}{s_m} e_m$$

for every $w \in W$, where T^* is the conjugate transpose and $T^\dagger$ is the pseudoinverse of T.

Exercise 11 (Matrix Multiplication (*))
Which of the following statements about matrix multiplication is true?

a Matrix multiplication is commutative.

b Matrix multiplication is associative.

c Matrix multiplication is distributive over addition.

d Both (b) and (c).

Exercise 12 (Rank of a Matrix (*))
Let A be a 3×3 matrix with rank 2. Which of the following statements is true?

a A has a non-trivial null space.

b A is invertible.

c The columns of A form a linearly independent set.

d $\det(A) \neq 0$.

Exercise 13 (Positive Definiteness (*))
True or False: A symmetric matrix with all positive eigenvalues is positive definite.

2 Calculus

Exercise 14 (Cauchy Sequences (*))
Prove that the sequence $x_n = (1/n, 1/n^2) \in \mathbb{R}^2$ is a Cauchy sequence. What is the accumulation point?
Hint: A higher dimensional space requires a norm (e.g. Euclidean norm) as a distance measure instead of the absolute value.

Exercise 15 (Sequences (*))
Given the sequence a_n:

$$a_n = (-1)^n$$

1. Determine all accumulation points of the sequence, if they exist.
2. Create another sequence b_n by changing one symbol of the sequence a_n, such that b_n has only one accumulation point. (For reference, a_n has 5 symbols in total.)
3. Create another sequence c_n by changing one symbol of the sequence a_n, such that b_n has no accumulation point.
4. Determine for a_n, b_n and c_n if the sequences converge. If a sequence converges, determine what it converges to. If it doesn't converge, explain where the definition of convergence breaks down.

Exercise 16 (Higher order derivatives (**))

Consider $f : \mathbb{R}^3 \to \mathbb{R}$ with $f(x) = x_1^2 x_2 + x_2 x_3^2 + e^{x_1+x_2+x_3}$ with $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3$.

1. Calculate the gradient.
2. Calculate the Hessian matrix.

Exercise 17 (Continuity and Differentiability (***))
Let $\xi = (0, 0)$ and

$$f : \mathbb{R}^2 \to \mathbb{R}, f(x, y) = \begin{cases} (x^2 + y^2) \sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) & (x, y) \neq \xi \\ 0 & (x, y) = \xi \end{cases}$$

Prove that f is continuous at ξ. Calculate $\frac{\partial f}{\partial x}$ at ξ and show that it is not continuous there. Show whether f is differentiable at ξ.

Exercise 18 (Partial derivatives (*))
Let $f(x, y) = x^2 y + 3xy^2 - y^3$. Compute the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Exercise 19 (Continuity (***))
Prove that a function $f : X \to Y$ is continuous at $x_0 \in X$ if and only if for every sequence $(x_n)_{n\in\mathbb{N}} \subset X$ with $x_n \to x_0$, we have $f(x_n) \to f(x_0)$.

Exercise 20 (Gradient and Critical Points (**))
Consider the function $f(x, y) = x^2 + y^2 - 4x - 6y + 13$. Find the critical points of $f(x, y)$ and determine their nature (local minima, maxima, or saddle points).

Exercise 21 (Differentiation and MVT (**))
Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ has partial derivatives at every point bounded by $A > 0$. Prove that there is an M such that:

$$|f((x, y)) - f((x_0, y_0))| \leq M\sqrt{(x - x_0)^2 + (y - y_0)^2}$$

Exercise 22 (Definite Integral (**))
Compute the following definite integral:

$$\int_0^{\pi} x \sin(x)\, dx.$$

3 Optimization

Exercise 23 (Convex Sets (*))

1. $$A_1 = \mathbb{R}^2, A_2 = \{x \in \mathbb{R}^2 | \|x\| <= 1\}, A_3 = \{x \in \mathbb{R}^2 | \|x\| = 1\}$$

   Pick every true statement:

   ☐ All Sets are convex
   ☐ $A_{1,2}$ are convex, A_3 is not convex
   ☐ A_1 is convex, $A_{2,3}$ is not convex
   ☐ None of the sets are convex

2. Pick every true statement:

   ☐ The union of convex sets is convex
   ☐ The intersection of convex sets is convex
   ☐ The union of non-convex sets is always convex
   ☐ The union of non-convex sets is always non-convex

Exercise 24 (Lagrangian (**))
Consider the constrained optimization problem:

$$\text{Minimize } f(x, y) = x^2 + y^2, \quad \text{subject to } x + 2y = 4$$

Exercise 25 (Lagrange Method (**))
Given $c \geq 0$, the following optimization problem is defined: Find $\max_{x,y\in\mathbb{R}} \log_2(x+1) + \log_2(y+1)$ subject to $x + y \leq 2$ and $x + 2y \leq c$.
Write down the corresponding Lagrangian and find the optimal solution (x, y) depending on c. Then, find the c for which the solution has the largest value and calculate that value.

Exercise 26 (Lagrange Multipliers (*))
Use the method of Lagrange multipliers to find the maximum and minimum values of $f(x, y) = x^2 + y^2$ subject to the constraint $g(x, y) = x + y - 1 = 0$.

Exercise 27 (Derivatives of Convex Functions (**))
Let $f : \mathbb{R} \to \mathbb{R}$ be convex and differentiable. Prove that f' is increasing, i.e. $x < y \implies f'(x) \leq f'(y)$.

Exercise 28 (Title (**))
Given a convex set C, suppose that x^* solves

$$\min_{x \in C} \frac{1}{2}\|x\|_2^2.$$

Show that for any $x \in C$,

$$\frac{1}{2}\|x - x^*\|_2^2 \leq \frac{1}{2}\|x\|_2^2 - \frac{1}{2}\|x^*\|_2^2.$$

Exercise 29 (Title (level of difficulty))
Let $f : \mathbb{R}^n \to \mathbb{R}$ and $f \in C^1$ be convex, and $\Omega = \{x \in \mathbb{R}^n : h(x) = 0\}$ where $h : \mathbb{R}^n \to \mathbb{R}^m$ such that Ω is convex and h is differentiable. Then $x^* \in \Omega$ is a global minimizer of f over Ω iff there exists $\lambda^* \in \mathbb{R}^m$ such that

$$\nabla f(x^*) + Dh(x^*)^\top \lambda^* = 0.$$