Winter term 2024/25
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Assignment 13 - Bonus

Mathematics for Machine Learning

Submission due Friday **31.01.25, 23:59** via Ilias
Assignment 13 will be the last assignment and consists only of bonus points. This means that you need at least 120 points in total to be admitted to the exam.

Justify all your claims.

Exercise 1 (**Maximum Likelihood Estimator**, 3+2 points).
Consider i.i.d. random variables $X_1, \dots, X_n \sim \mathcal{N}(\mu, \sigma^2)$ that are normally distributed with expected value $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$.

a) Assume that both μ and σ^2 are unknown. Calculate the Maximum Likelihood estimates $\hat{\mu} = \hat{\mu}(X_1, \dots, X_n)$ and $\hat{\sigma}^2 = \hat{\sigma}^2(X_1, \dots, X_n)$ for the expected value and the variance.

b) Now we add the restriction that $\mu \geq 0$. Again, calculate the Maximum Likelihood estimator $\hat{\mu}_{\geq 0} = \hat{\mu}_{\geq 0}(X_1, \dots, X_n)$ for the restricted case.

Exercise 2 (**Testing basics**, 1+1+2.5 points).
A new treatment for a disease is claimed to improve the recovery rate of patients. It is known that 20% of all patients recover without any treatment. We now observe for $n \in \mathbb{N}$ independent patients whether they recover after being treated:

$$X_i = \begin{cases} 1, & \text{if patient } i \text{ recovers} \\ 0, & \text{otherwise} \end{cases}, \quad i \in \{1, \dots, n\}.$$

a) Formalize this test setting by stating the distribution of the X_i, the null hypothesis H_0, the alternative hypothesis H_1 and the corresponding parameter spaces Θ_0 and Θ_1.

Now Let $x = (x_1, \dots, x_n) \in \{0,1\}^n$ be a sample. Now consider the following three tests $\varphi_j \colon \{0,1\}^n \to \{0,1\}$ to decide based on the sample, if we reject (for $\varphi(x) = 1$) or retain (for $\varphi(x) = 0$) the null hypothesis H_0:

$$\varphi_1(x) = x_1, \quad \varphi_2(x) = \prod_{i=1}^{n} x_i, \quad \varphi_3(x) = \begin{cases} 1, & \text{if } \sum_{i=1}^n x_i \geq m \\ 0, & \text{otherwise} \end{cases} \quad \text{for some } m \in \mathbb{N}_0$$

b) Reformulate each of the three tests in terms of a test statistic and a rejection region such that $\varphi_j(x_1, \dots, x_n) = \mathbb{1}_{\{T_j(x_1, \dots, x_n) \in R_j\}}$ for $j \in \{1, 2, 3\}$.

c) Compute power function and level for each test. Additionally, state the level of φ_3 for $n = 10$ and $m = 5$.
Hint: You may use that $\sum_{k=m}^{n} \binom{n}{k} \theta^k (1-\theta)^{n-k}$ *is increasing in* θ.

Exercise 3 (**Two-tailed testing**, 2.5+1+2 points)**.**
Consider i.i.d. samples from a normal distribution $X_1, \dots, X_n \sim \mathcal{N}(\mu, \sigma^2)$ with $\sigma > 0$ known. We want to test the hypothesis $H_0 : \mu = 0$ against $H_1 : \mu \neq 0$. Consider the test with test statistic $T(x_1, \dots, x_n) = \left|\frac{1}{n}\sum_{i=1}^n x_i\right|$ and rejection region $R = [c, \infty)$ for some $c > 0$.

a) Show that the power function $\beta(\mu)$ satisfies

$$\beta(\mu) = 2 - \Phi\left(\frac{\sqrt{n}}{\sigma}(c-\mu)\right) - \Phi\left(\frac{\sqrt{n}}{\sigma}(c+\mu)\right),$$

where Φ is the cumulative distribution function of a standard normal distribution, that is, $\Phi(x) = P(Z \leq x)$ for $Z \sim \mathcal{N}(0,1)$ and $x \in \mathbb{R}$.
Hint: Use the following three properties of normal distributions:

$$\begin{aligned} X \sim \mathcal{N}(\mu_X, \sigma_X^2),\ Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2) \text{ independent} \quad &\Rightarrow \quad X + Y \sim \mathcal{N}(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2) \\ X \sim \mathcal{N}(\mu_X, \sigma_X^2) \text{ and } a, b \in \mathbb{R} \quad &\Rightarrow \quad aX + b \sim \mathcal{N}(a\mu_X + b, a^2\sigma_X^2) \\ \Phi(-x) &= 1 - \Phi(x) \quad \forall x \in \mathbb{R}\,. \end{aligned}$$

b) Let $\alpha \in (0,1)$. Show that the value $c = c_\alpha$, which leads to a test of level α, is given by

$$c_\alpha = \frac{\sigma}{\sqrt{n}}\Phi^{-1}\left(1 - \frac{\alpha}{2}\right).$$

c) Show that the p-value of a sample $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ is given by

$$p(x) = 2\left(1 - \Phi\left(\frac{\sqrt{n}}{\sigma}T(x)\right)\right).$$

Hint: $\alpha \mapsto c_\alpha$ is decreasing in α.

Exercise 4 (**Own Exam Question**, 4 points)**.**
Design your own exam questions! This is a good way to recap and understand the concepts discussed so far. You may hand in LaTeX-source code of up to 2 self-created exam questions, one about probability theory and one about statistics (as far as we have gotten in the lecture). Mark them with a level of difficulty from (*) (easy), (**) (medium) and (***) (hard). Each exam question gives you 2 bonus points if

- they are mathematically correct,
- they do not simply ask to reproduce a definition or a theorem from the lecture. We won't ask questions like this as you are allowed to bring a cheat sheet to the exam,
- they are solvable within a time span typical for an exam (5-15 minutes),
- your LaTeX-code compiles.

We will collect all the questions and upload them for everybody to have some material to prepare for the exam with.

On ILIAS, within all the assignment submissions you find the separate unit "Students' Exam Questions". There, you can download our LaTeX-template and fill in your questions. You also upload your questions there, separate from the rest of this assignment, as a .tex-file (not as a pdf!).