Winter term 2024/25
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# Assignment 10

## Mathematics for Machine Learning

Submission due Friday **10.01.2025, 23:59** via Ilias

Justify all your claims.

**Exercise 1** (**Convolutions**, 1+2+2 points)**.**

a) Let $X, Y$ be two independent discrete random variables and set $Z := X + Y$. Prove that the probability distribution of $Z$ is given by

$$P(Z = z) = \sum_{x \in \mathrm{im}(X)} P(X = x) \cdot P(Y = z - x).$$

b) Show that the sum of two independent Poisson-distributed random variables is again Poisson-distributed.

c) For independent continuous random variables $X, Y$ with $Z = X + Y$ it similarly holds that the density $p_Z$ of $Z$ is given by

$$p_Z(z) = \int_{-\infty}^{\infty} p_X(x) \cdot p_Y(z - x) \mathrm{d}x,$$

where $p_X$ and $p_Y$ are the densities of $X$ and $Y$ respectively. Use this result to prove that the sum of two independent random variables $X, Y \sim \mathcal{N}(0, 1)$ is again normally distributed. You don't not need to prove the formula for $p_Z$.
*Remark: This also holds true for independent normally distributed random variables with arbitrary mean and variance, but the proof is much more tedious.*

**Exercise 2** (**Weak law of large numbers**, 4 points)**.**
Let $X_1, X_2, ...$ be a sequence of i.i.d. random variables where $X_i \in \mathcal{L}^2(\Omega, \mathcal{A}, P)$ and define $S_n := \sum_{i=1}^{n} X_i$. Prove

$$\frac{1}{n} S_n \xrightarrow{P} \mathrm{E}(\mathrm{X}_1).$$

You may not use the strong law of large numbers.

**Exercise 3** (**Poisson distribution**, 3+2+1 points)**.**

a) Let $X$ be Poisson-distributed with parameter $\lambda > 0$. Compute $\mathrm{E}(X)$ and $\mathrm{Var}(X)$.
*Hint: For the variance, use the identity $k^2 = k(k-1) + k$.*

b) Let $\lambda > 0$ and $X_n \sim \mathrm{Bin}(n, \lambda/n)$ for $n \in \mathbb{N}$. Prove that for all $k \in \mathbb{N}$ it holds

$$P(X_n = k) \xrightarrow{n \to \infty} P(X = k),$$

where $X$ follows a Poisson-distribution with parameter $\lambda$.
*Hint: You may use $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = \mathrm{e}^x$.*

c) What is the interpretation of the result in b)? What scenarios can be modeled by a Poisson distribution?

**Exercise 4** (**Convergence of random variables**, 5 points).
Let $X, X_1, X_2, ...$ be real-valued random variables on some probability space $(\Omega, \mathcal{P}(\Omega), P)$. Prove that if $\Omega$ is countable, then $X_n \xrightarrow{P} X$ implies $X_n \xrightarrow{a.s.} X$.
*Hint: For any $\varepsilon > 0$ and $\omega \in \Omega$ with $P(\omega) > 0$, prove and use that there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ it holds $P(|X_n - X| > \varepsilon) < P(\omega)$.*

**Exercise 5** (**Bonus exercise**, up to 6 points).
Design your own exam questions! This is a good way to recap and understand the concepts discussed so far. You may hand in LaTeX-source code of up to 3 self-created exam questions, one about linear algebra, one about calculus and one about optimization. Mark them with a level of difficulty from (*) (easy), (**) (medium) and (***) (hard). Each exam question gives you 2 bonus points if

- they are mathematically correct,
- they do not simply ask to reproduce a definition or a theorem from the lecture. We won't ask questions like this as you are allowed to bring a cheat sheet to the exam,
- they are solvable within a time span typical for an exam (5-15 minutes),
- your LaTeX-code compiles.

We will collect all the questions and upload them for everybody to have some material to prepare for the exam with.

On ILIAS, within all the assignment submissions you find the separate unit "Students' Exam Questions". There, you can download our LaTeX-template and fill in your questions. You also upload your questions there, separate from the rest of this assignment, as a .tex-file (not as a pdf!).