Winter term 2024/25
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Assignment 05

Mathematics for Machine Learning

Submission due Friday **22.11.2024, 23:59** via Ilias

Justify all your claims.

Exercise 1 (**Recursive sequences**, 2+2 points)**.**

a) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. Prove that if $(a_n)_{n\in\mathbb{N}}$ is monotone increasing (that is $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$) and has an upper bound, then it converges to its supremum.

b) Prove that the recursive sequence $a_{n+1} = \sqrt{a_n + 2}$ with $a_0 = 0$ converges and determine its limit.

Exercise 2 (**Continuity**, 1+1+2 points)**.**

a) Prove that every Lipschitz continuous function $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous.

b) Prove that $g : \mathbb{R} \to \mathbb{R},\ x \mapsto x^2$ is not uniformly continuous.

c) Prove that $h : \mathbb{R}_{\geq 0} \to \mathbb{R},\ x \mapsto \sqrt{x}$ is uniformly continuous but not Lipschitz continuous.

Exercise 3 (**Uniform convergence**, 2+1+1+2 points)**.**

a) Analyze whether the following sequences of functions converge pointwise. If they do, state the limit and prove whether they converge uniformly.

$$
\begin{aligned}
&(f_n)_{n\in\mathbb{N}} \quad \text{with} \quad f_n\colon \mathbb{R} \to \mathbb{R}, \quad x \mapsto \frac{1}{n}\sin(nx) \\
&(g_n)_{n\in\mathbb{N}} \quad \text{with} \quad g_n\colon \mathbb{R} \to \mathbb{R}, \quad x \mapsto x + \frac{x}{n}\cos(x)
\end{aligned}
$$

b) Consider a sequence of functions $f_n\colon \mathcal{D} \to \mathbb{R}$ on a finite set $\mathcal{D}$ that converges pointwise to a function $f\colon \mathcal{D} \to \mathbb{R}$. Prove that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f.

Consider a sequence of functions $f_n\colon [a,b] \to \mathbb{R}$ for $n \in \mathbb{N}$, which are all Lipschitz continuous with the same Lipschitz constant $L > 0$. Assume that this sequence converges pointwise to $f\colon [a,b] \to \mathbb{R}$, where $a, b \in \mathbb{R}$ and $a < b$.

c) Prove that f is also Lipschitz continuous with Lipschitz constant L.

d) Prove that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f.
Hint: For an $\varepsilon > 0$ consider the set $\mathcal{D}$ of points $a, a+\varepsilon, a+2\varepsilon, \ldots$ up to b. Then let $\lfloor x \rfloor := \max\{y \in \mathcal{D} \mid y \leq x\}$ and use the equality

$$
f_n(x) - f(x) = f_n(x) - f_n(\lfloor x \rfloor) + f_n(\lfloor x \rfloor) - f(\lfloor x \rfloor) + f(\lfloor x \rfloor) - f(x)
$$

Exercise 4 (**Power and Taylor series**, 2+2+2 points).

a) Determine the radius of convergence of the power series

$$\sum_{j=1}^{\infty} \frac{j^2}{2^j} x^j \quad \text{and} \quad \sum_{j=1}^{\infty} 3^j x^{j^2}.$$

b) Compute the Taylor polynomial of $f\colon \mathbb{R}_{\geq 0} \to \mathbb{R}, x \mapsto e^{\pi - x} \sin x$ in $a = \pi$ of degree $n = 3$ and the corresponding Lagrange remainder. Find an upper bound for the remainder by bounding $\sup_{\xi \geq 0} f^{(4)}(\xi)$ with a constant.

c) Prove that f from part b) is equal to its Taylor series, that is, $f(x) = \lim_{n\to\infty} T_n(x, \pi)$ for all $x \in \mathbb{R}_{\geq 0}$.
Hint What is the connection between f and $f^{(4)}$?