Winter term 2024/25
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Assignment 01

Mathematics for Machine Learning

Submission due on Friday, **25.10.2024, 12:00** via Ilias

Justify all your claims.

Exercise 1 (**Dictionary Learning**, 1 + 1 + 2 + 3 points)**.**
In this exercise, you will explore the concept of dictionary learning, where the goal is to find a set of vectors D (a "dictionary") such that data points can be represented as sparse linear combinations of these vectors. For further reading, you can look up "dictionary learning" or "compressed sensing".

1. Consider a vector $v = (2, 1, 1) \in \mathbb{R}^3$ and a dictionary D consisting of the following five vectors:

$$D = \{d_1 = (1, 0, 0), d_2 = (0, 1, 0), d_3 = (0, 0, 1), d_4 = (1, 1, 0), d_5 = (0, 1, 1)\}.$$

   Express v as a linear combination of

   i) exactly three dictionary vectors.
   ii) exactly four dictionary vectors.
   iii) all five dictionary vectors.

2. Consider a vector $v \in \mathbb{R}^3$ and a dictionary D consisting of five vectors spanning $\mathbb{R}^3$. Can v be written in multiple ways as a linear combination of the dictionary vectors? If yes, provide a proof. If no, give a counterexample.

3. Find a dictionary in $\mathbb{R}^3$ with a minimal number of vectors, such that each of the following points can be expressed as a linear combination of exactly two dictionary vectors.

   i) $v_1 = (1, 1, 1), \quad v_2 = (2, 1, 1), \quad v_3 = (1, 2, 2)$
   ii) $v_1 = (1, 1, 1), \quad v_2 = (2, 1, 0), \quad v_3 = (1, 2, 0)$

4. Assume you have n arbitrary data points in $\mathbb{R}^3$. Let D be a dictionary of vectors. Can you give upper and lower bounds on the number of dictionary vectors that are necessary such that each point can be spanned by exactly

   i) one vector,
   ii) two vectors,
   iii) three vectors.

Exercise 2 (**Basis and Dimension**, 4 points)**.**
Let V be a finite-dimensional vector space with basis $\mathcal{B} = (v_1, \ldots, v_n)$. Consider a vector $v = \sum_{i=1}^{n} \lambda_i v_i$ with $\lambda_k \neq 0$ for $1 \leq k \leq n$.
Prove that $\tilde{\mathcal{B}} = (v_1, \ldots, v_{k-1}, v, v_{k+1}, \ldots, v_n)$ is also a basis of V.

Exercise 3 (**Linear Mappings and Vector Spaces**, 2 + 3 points)**.**
Let V and W be two vector spaces over a field F. Consider the set of linear mappings $\mathcal{L}(V, W)$ with the operations of addition and scalar multiplication defined by

$$(S + T)(v) := Sv + Tv \text{ and } (\lambda T)(v) := \lambda(Tv)$$

for $S, T \in \mathcal{L}(V, W)$ and for all $v \in V$.

a) Verify that $S + T$ and λT are again linear maps in $\mathcal{L}(V, W)$.

b) Prove that $\mathcal{L}(V, W)$ with the above operations is a vector space.

Exercise 4 (**Linear Mappings**, 2 + 2 points)**.**
Suppose $v_1, \ldots, v_n$ is a list of vectors in some vector space V. Define $T \in \mathcal{L}(\mathbb{R}^n, V)$ by $T(\lambda_1, \ldots, \lambda_n) = \lambda_1 v_y + \cdots + \lambda_n v_n$.

a) Which property does T need to have such that $v_1, \ldots, v_n$ span V?

b) What property does T need to have such that $v_1, \ldots, v_n$ are linearly independent?